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ABSTRACT

A rotating ensemble of bodies with angular periodicity rotates in the presence of a similar, stationary ensemble of generally different periodicity. The structure is illuminated from an electromagnetic source. The scattered field is modulated periodically as the scatterer rotates, and contains a discrete spectrum of frequency components.

The scattered spectrum is predicted through electromagnetic field theory. The theory has been developed such as to exploit the angular periodicities of the ensembles and thereby reduce the computational load by a considerable factor.

The spectrum consists of lines separated by the "body rate", i.e. the rate of rotation multiplied by the number of bodies in the rotating ensemble The total bandwidth is several times greater than that for the rotating ensemble in free-space, due to electromagnetic interaction between the two ensembles.

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The scattered spectrum is predicted through electromagnetic field theory. The theory has been developed such as to exploit the angular periodicities of the ensembles and thereby reduce the computational load by a considerable factor.

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1. Introduction

The radar scatter from jet engines is periodically modulated by their rotation, resulting in a line spectrum with features characteristic of the engine. Two previous studies [1,2] of the radar scatter from rotating bodies with angular periodicity have been undertaken towards predicting the spectrum.

In the first study, a single stage of the engine compressor was modelled as an ensemble of coplanar wires radiating from an axis. Conclusions were drawn concerning the width of the spectrum and the line separation, and predictions were attempted of the effects of other features of engine compressors, such as fan blades rather than wires, multiple stages, the engine intake, stator stages etc.

In the second study, a limitation of the wire model was addressed: the wires were replaced by an ensemble of bodies with arbitrary shape but angular periodicity. This extension confirmed that many of the earlier conclusions remained valid for a single stage of the engine, i.e. for a set of blades mounted on a central hub.

The present work extends the model further by including a stationary structure in the presence of which the original structure rotates. The stationary structure is an ensemble of conducting bodies with angular periodicity, coaxial with the rotating ensemble. The stationary ensemble may be used to model one set of stator blades in the engine, or a circular engine cowling, or both.

In both the earlier studies the incident field was represented as a spacial spectrum of harmonics of the azimuthal angle ϕ measured about the z axis. Through this device it is found necessary, for each harmonic, to determine the current distribution on one body only, for one azimuthal angle of incidence at one time; all other cases are related to the first through a phase shift. This current is the solution of an electric field integral equation [3] for scattering from a single body.

The spacial harmonic technique is again employed, leading to a large reduction in computing storage and time.

The approach taken here involves the development of a dyadic Green's function for the stationary ensemble. This builds on the work in [2] but requires determination of the scattered field in the more general case where the source and observation points are in the neighbourhood of the scatterer, whereas in the earlier work these points were remote.

The field scattered by the rotating ensemble is determined with the help of the Green's function for the stationary ensemble. The field incident, in the presence of the stationary ensemble, on the rotating ensemble, is written in terms of this Green's function. Periodicities in the geometry are exploited to reduce the computation of the currents on the rotating bodies. These currents radiate the scattered field in the presence of the stationary ensemble; the field is calculated with the Green's function for the stationary ensemble.

As has been a feature of the previous studies, certain key properties of the scattered field are apparent from the mathematical form of the equations without need for their numerical solution, and attention is drawn to these. Notably, the scattered field spectrum consists of lines separated by the "body rate", ie. the product of the rotation rate and the number of bodies in the

rotating ensemble. No new frequency components are introduced by the presence of the stationary ensemble. There is an extension of the width of the spectrum, beyond the limits seen for the body rotating in free space, which is attributable to new forms of electromagnetic interaction between stationary and moving bodies.

The results presented here for two angularly periodic ensembles of bodies with relative rotation are valid for the limiting case where one or both ensembles contain only one body; thus in particular, the field scattered from a periodic ensemble rotating in the presence of an arbitrary stationary body will have a spectrum with lines separated by the body rate, and of bandwidth greater than that for the ensemble rotating in free space.

2. The stationary ensemble

In this section is described the geometry of the ensemble of conducting bodies, together with the coordinate systems and some special parameters to be used below.

An ensemble S of I conducting bodies, with surfaces S_i, $i=0,1,\ldots,I-1$, is arranged about an axis (in the coordinate system to be used, the z axis), see figure 2.1. The surfaces are the similar and oriented such that S_i is generated by rotating S_0 about the z axis through the azimuthal angle

$$\Phi_i = i\frac{2\pi}{I}. \tag{2.1}$$

The bodies may be detached, as shown, or attached to one another. In the latter case the individual bodies must be defined by arbitrary cuts through the common region, creating an ensemble of bodies. This is illustrated in figure 2.2 which is suggestive of the stator blades of an engine, fixed to the cowling; the dotted lines are the cuts.

Rectangular, cylindrical and spherical coordinates will be used as convenient; the system in use will usually be apparent from the conventional symbols: (x,y,z), (ρ,ϕ,z), (r,θ,ϕ).

If a point on surface S_0 has position vector and coordinates

$$\bar{r}_0 \equiv (x_0,y_0,z_0) \equiv (\rho_0,\phi_0,z_0) \equiv (r_0,\theta_0,\phi_0),$$

with the conventional relations

$$\begin{aligned} \rho_0 &= (x_0^2+y_0^2)^{\frac{1}{2}},\ \phi_0 = \cos^{-1}\frac{x_0}{\rho_0} = \sin^{-1}\frac{y_0}{\rho_0}, \\ r_0 &= (x_0^2+y_0^2+z_0^2)^{\frac{1}{2}},\ \theta_0 = \cos^{-1}\frac{z_0}{r_0} = \sin^{-1}\frac{\rho_0}{r_0} \end{aligned} \tag{2.2}$$

then a point of position vector

$$\bar{r}_i \equiv (\rho_i,\phi_i,z_i) \equiv (r_i,\theta_i,\phi_i)$$

with

$$\rho_i = \rho_0,\ \phi_i = \phi_0 + \Phi_i,\ z_i = z_0,\ r_i = r_0,\ \theta_i = \theta_0 \tag{2.3}$$

lies on S_i.

Points related like $\bar{r}_0$ and $\bar{r}_i$ will be called "corresponding points". Such points are illustrated in figure 2.3. Wherever $\bar{r}_0$ and $\bar{r}_i$ or their similarly subscripted coordinates appear in the same equation, they are to be understood as corresponding in this sense.

The relation between $\bar{r}_0$ and $\bar{r}_i$ may be written in the form

$$\bar{r}_i = \bar{\bar{R}}_i . \bar{r}_0 \tag{2.4}$$

where $\bar{\bar{R}}_i$ is a dyadic which rotates a vector about the z axis through angle Φ_i. In rectangular coordinates (2.4) corresponds to the matrix equation

$$\begin{bmatrix} r_{ix} \\ r_{iy} \\ r_{iz} \end{bmatrix} = \begin{bmatrix} \cos\Phi_i & -\sin\Phi_i & 0 \\ \sin\Phi_i & \cos\Phi_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{0x} \\ r_{0y} \\ r_{0z} \end{bmatrix} \tag{2.5}$$

where r_{ix} etc are the rectangular components of $\bar{r}_i$.

$\bar{\bar{R}}_i$ may be expanded in terms of spherical unit vectors thus:

$$\bar{\bar{R}}_i = \hat{r}_i \hat{r}_0 + \hat{\theta}_i \hat{\theta}_0 + \hat{\phi}_i \hat{\phi}_0 \tag{2.6}$$

where the unit vectors subscripted i and 0 are evaluated at $\bar{r}_i$ and $\bar{r}_0$ respectively.

It is convenient to define a more general rotational dyadic $\bar{\bar{R}}(\Delta\phi)$ which rotates a vector about the z axis through azimuthal angle $\Delta\phi$. The dyadic $\bar{\bar{R}}_i$ is the special case where $\Delta\phi = \Phi_i$.

Note that the dyadic $\bar{\bar{R}}^{-1}(\Delta\phi)$, which reverses the effect of $\bar{\bar{R}}(\Delta\phi)$, is related to it by

$$\bar{\bar{R}}^{-1}(\Delta\phi) = \bar{\bar{R}}(-\Delta\phi) \quad . \tag{2.7}$$

We remark that any vector function of position $\bar{c}(\bar{r})$, whose scalar components expressed in spherical coordinates are not functions of ϕ, takes values at two positions $\bar{r}_1$ and $\bar{r}_2$ related by

$$\bar{c}(\bar{r}_2) = \bar{\bar{R}}(\Delta\phi) . \bar{c}(\bar{r}_1) \quad . \tag{2.8}$$

if $\bar{r}_1$ and $\bar{r}_2$ are related by $\bar{r}_2 = \bar{\bar{R}}(\Delta\phi) . \bar{r}_1$.

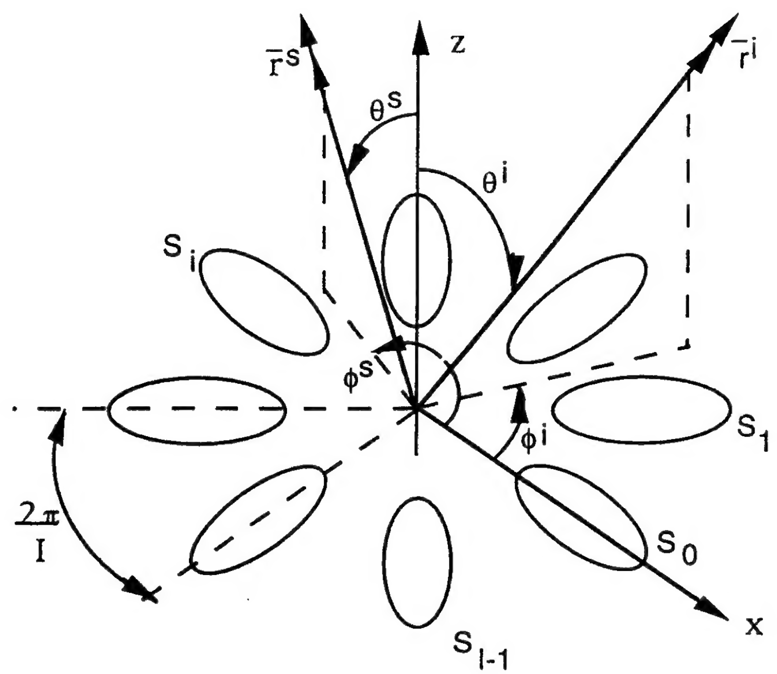

Figure 2.1: The ensemble of bodies with surfaces S_0, S_1,...S_{I-1}, disposed with angular periodicity about the z axis. A plane wave is incident from the direction of spherical coordinates θ^i,ϕ^i. The scattered field is observed from the direction θ^s,ϕ^s.

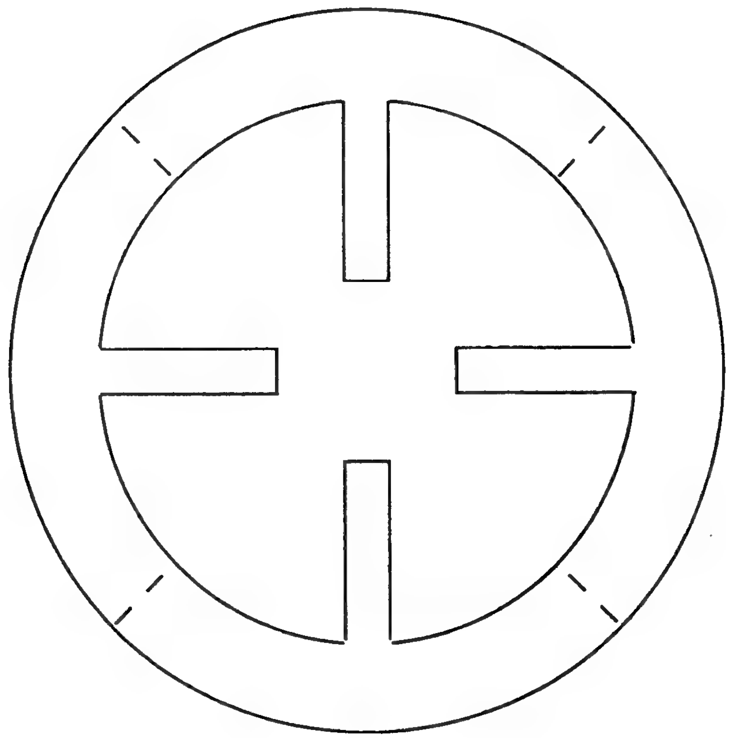

Figure 2.2: A single body with angular periodicity, resolved into an ensemble of (contiguous) bodies by arbitrarily, but periodically, placed cuts in the positions of the dashed lines.

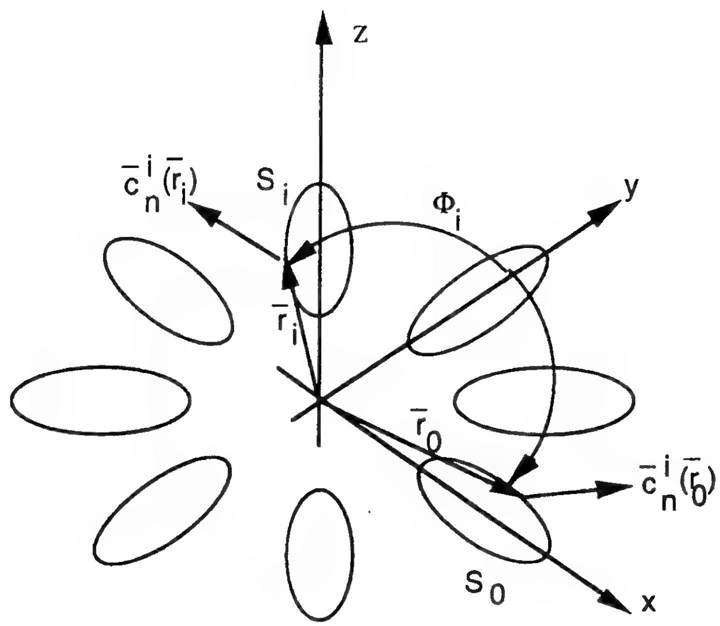

Figure 2.3: "Corresponding" points $\bar{r}_0$ and $\bar{r}_i$ on bodies S_0 and S_i. The vectors $\bar{c}^i_n(\bar{r}_0)$ and $\bar{c}^i_n(\bar{r}_i)$ are rotated relatively to one another about the z axis in the same way as $\bar{r}_0$ and $\bar{r}_i$.

3. The free-space dyadic Green's function

3.1 A direct derivation from the dipole field

The electric field $\overline{E}$ at a point of position vector $\overline{r}$ of an ideal dipole of current moment $\overline{p}$ at a point of position vector $\overline{r}'$ may be written in terms of the free-space dyadic Green's function $\overline{\overline{G}}^0(\overline{r},\overline{r}')$, thus:

$$\overline{E}(\overline{r}) = \overline{\overline{G}}^0(\overline{r},\overline{r}').\overline{p} \tag{3.1}$$

Through manipulation of the formula for the field of an ideal dipole [e.g. 3] , $\overline{\overline{G}}^0(\overline{r},\overline{r}')$ is found to take the form

$$\overline{\overline{G}}^0(\overline{r},\overline{r}') = G'\,\overline{\overline{I}} + G''\,\overline{RR} \tag{3.2}$$

where

$$G'(R) = \frac{1}{j\omega\varepsilon}\frac{1}{4\pi R^3}(-1 - jk_0R + k_0^2R^2)e^{-jk_0R} \quad , \tag{3.3}$$

$$G''(R) = \frac{1}{j\omega\varepsilon}\frac{1}{4\pi R^5}(3 + j3k_0R - k_0^2R^2)e^{-jk_0R} \quad , \tag{3.4}$$

$$\overline{R} = \overline{r} - \overline{r}' \tag{3.5}$$

and k_0 is the wave number for $e^{j\omega_0 t}$ time dependence. The time dependence is here suppressed, as is conventional. It will be reintroduced when convenient.

The field of a current density distribution $\overline{J}(\overline{r})$ is then

$$\overline{E}(\overline{r}) = \iiint_V \overline{\overline{G}}^0(\overline{r},\overline{r}').\overline{J}(\overline{r}')dv' \tag{3.6}$$

in which the volume V is that containing the current.

3.2 A Green's function in terms of spherical harmonics

A formula for $\overline{\overline{G}}^{0}(\bar{r},\bar{r}')$ as a summation of spherical harmonics is given by [4]:

$$\overline{\overline{G}}^{0}(\bar{r},\bar{r}') = -\frac{jk_0}{4\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{n}(2-\delta_m)\frac{2n+1}{n(n+1)}\frac{(n-m)!}{(n+m)!}\sum_{e,o}$$

$$\begin{cases} \overline{M}^{(4)}_{\substack{e\\o}mn}\ \overline{M}^{(1)'}_{\substack{e\\o}mn} + \overline{N}^{(4)}_{\substack{e\\o}mn}\ \overline{N}^{(1)'}_{\substack{e\\o}mn}\ , & r > r' \\ \overline{M}^{(4)}_{\substack{e\\o}mn}\ \overline{M}^{(1)'}_{\substack{e\\o}mn} + \overline{N}^{(4)}_{\substack{e\\o}mn}\ \overline{N}^{(1)'}_{\substack{e\\o}mn}\ , & r < r' \end{cases} \tag{3.7}$$

where

$$\delta_m = \begin{cases} 1\,, & m = 0 \\ 0\,, & m \neq 0 \end{cases}\quad , \tag{3.8}$$

$$\overline{M}^{(i)}_{\substack{e\\o}mn}(\bar{r}) = \mp\frac{m}{\sin\theta}z^{(i)}_n P^m_n \begin{matrix}\sin\\ \cos\end{matrix} m\phi\,\hat{\theta} - z^{(i)}_n \dot{P}^m_n \begin{matrix}\cos\\ \sin\end{matrix} m\phi\,\hat{\phi}\quad , \tag{3.9}$$

$$\begin{aligned}\overline{N}^{(i)}_{\substack{e\\o}mn}(\bar{r}) = {} & \frac{n(n+1)}{k_0 r}z^{(i)}_n P^m_n \begin{matrix}\cos\\ \sin\end{matrix} m\phi\,\hat{r} \\ & + \dot{z}^{(i)}_n(\dot{P}^m_n \begin{matrix}\cos\\ \sin\end{matrix} m\phi\,\hat{\theta} \mp \frac{m}{\sin\theta}P^m_n \begin{matrix}\sin\\ \cos\end{matrix} m\phi\,\hat{\phi})\end{aligned}\quad , \tag{3.10}$$

the primes on these functions specify that they are to be evaluated at $\bar{r}'$,

$$z^{(1)}_n = j_n(k_0 r)\,,\ z^{(4)}_n = h^{(2)}_n(k_0 r)\quad , \tag{3.11}$$

(the spherical Bessel function and Hankel function of the second kind [5]),

$$\dot{z}^{(i)}_n = \frac{1}{k_0 r}\frac{\partial}{\partial r}(r z^{(i)}_n)\ , \tag{3.12}$$

$$P^m_n = P^m_n(\cos\theta)\quad , \tag{3.13}$$

(the associated Legendre function [5]) and

$$\dot{P}^m_n = \frac{\partial P^m_n}{\partial\theta}\quad . \tag{3.14}$$

In the above, the formulas of [4] have been changed for consistency with the present notation and the time dependence $e^{j\omega t}$ (vice $e^{-i\omega t}$).

If the substitutions

$$\sin m\phi = \frac{(e^{jm\phi} - e^{-jm\phi})}{2j}, \quad \cos m\phi = \frac{(e^{jm\phi} + e^{-jm\phi})}{2} \tag{3.15}$$

are made in (3.9) and (3.10) and the results are substituted into (3.7), one finds, after considerable algebraic manipulation

$$\overline{\overline{G}}^0(\bar{r},\bar{r}') = -\frac{jk_0}{4\pi}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{2n+1}{n(n+1)} \begin{cases} \overline{m}_{mn}^{(4)}\overline{m}_{-mn}^{(1)'} + \overline{n}_{mn}^{(4)}\overline{n}_{-mn}^{(1)'}, & r > r' \\ \overline{m}_{mn}^{(4)}\overline{m}_{-mn}^{(1)'} + \overline{n}_{mn}^{(4)}\overline{n}_{-mn}^{(1)'}, & r < r' \end{cases} \tag{3.16}$$

where

$$\overline{m}_{mn}^{(i)}(\bar{r}) = z_n^{(i)}\left(j\frac{m}{\sin\theta}P_n^m\,\hat{\theta} - \dot{P}_n^m\,\hat{\phi}\right)e^{jm\phi} \tag{3.17}$$

$$\overline{n}_{mn}^{(i)}(\bar{r}) = \left(\frac{n(n+1)}{k_0 r}z_n^{(i)}P_n^m\,\hat{r} + \dot{z}_n^{(i)}\left(\dot{P}_n^m\,\hat{\theta} + j\frac{m}{\sin\theta}P_n^m\,\hat{\phi}\right)\right)e^{jm\phi} \tag{3.18}$$

Upon substituting (3.17) and (3.18) into (3.16), and using the relation

$$P_n^{-m} = \frac{(n-m)!}{(n+m)!}P_n^m \tag{3.19}$$

one finds a further expression for the dyadic Green's function:

$$\overline{\overline{G}}^0(\bar{r},\bar{r}') = -\frac{jk_0}{4\pi}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{2n+1}{n(n+1)}\frac{(n-m)!}{(n+m)!}\overline{\overline{g}}_{mn}(\bar{r},\bar{r}')e^{jm(\phi-\phi')} \tag{3.20}$$

where

$$\begin{aligned} \overline{\overline{g}}_{mn} = {} & g_{rr'}\hat{r}\hat{r}' + g_{r\theta'}\hat{r}\hat{\theta}' + g_{r\phi'}\hat{r}\hat{\phi}' + g_{\theta r'}\hat{\theta}\hat{r}' + g_{\theta\theta'}\hat{\theta}\hat{\theta}' + g_{\theta\phi'}\hat{\theta}\hat{\phi}' \\ & + g_{\phi r'}\hat{\phi}\hat{r}' + g_{\phi\theta'}\hat{\phi}\hat{\theta}' + g_{\phi\phi'}\hat{\phi}\hat{\phi}' \end{aligned} \tag{3.21}$$

and for $r > r'$,

$$g_{rr'} = \frac{n^2(n+1)^2}{k_0^2 rr'}h_n j_n{}' P_n^m P_n^{m\prime}$$

$$g_{r\theta'} = \frac{n(n+1)}{k_0 r} h_n \dot{j}_n' P_n^m \dot{P}_n^{m'}$$

$$g_{r\phi'} = -j\frac{n(n+1)}{k_0 r} h_n \dot{j}_n' \frac{m}{\sin\theta'} P_n^m P_n^{m'}$$

$$g_{\theta r'} = \frac{n(n+1)}{k_0 r'} \dot{h}_n j_n' \dot{P}_n^m P_n^{m'}$$

$$g_{\theta\theta'} = h_n j_n' \frac{m^2}{\sin\theta\sin\theta'} P_n^m P_n^{m'} + \dot{h}_n \dot{j}_n' \dot{P}_n^m \dot{P}_n^{m'}$$

$$g_{\theta\phi'} = -j(h_n j_n' \frac{m}{\sin\theta} P_n^m \dot{P}_n^{m'} + \dot{h}_n \dot{j}_n' \frac{m}{\sin\theta'} \dot{P}_n^m P_n^{m'})$$

$$g_{\phi r'} = j\frac{n(n+1)}{k_0 r'} \dot{h}_n j_n' \frac{m}{\sin\theta} P_n^m P_n^{m'}$$

$$g_{\phi\theta'} = j(h_n j_n' \frac{m}{\sin\theta'} \dot{P}_n^m P_n^{m'} + \dot{h}_n \dot{j}_n' \frac{m}{\sin\theta} P_n^m \dot{P}_n^{m'})$$

$$g_{\phi\phi'} = h_n j_n' P_n^m P_n^{m'} + \dot{h}_n \dot{j}_n' \frac{m^2}{\sin\theta\sin\theta'} P_n^m P_n^{m'} \tag{3.22}$$

while for $r < r'$ the elements $g_{rr'}$, $g_{r\theta'}$ etc are found through the exchange of h_n and j_n.

(3.20) may be further rearranged into the form

$$\overline{\overline{G}}^0(\bar{r},\bar{r}') = \sum_{m=-\infty}^{\infty} \overline{\overline{G}}_m^0(\bar{r},\bar{r}') e^{jm(\phi-\phi')} \tag{3.23}$$

where

$$\overline{\overline{G}}_m^0(\bar{r},\bar{r}') = -\frac{jk_0}{4\pi} \sum_{n=|m|,\neq 0}^{\infty} \frac{2n+1}{n(n+1)} \frac{(n-m)!}{(n+m)!} \overline{\overline{g}}_{mn}(\bar{r},\bar{r}') \tag{3.24}$$

3.3 Remote source

(3.24) might be specialised for this case by substituting for $h_n^{(2)}(k_0 r')$ where it occurs in the elements of $\overline{\overline{g}}_{mn}(\overline{r}_0, \overline{r}')$, see (3.22), the large argument approximation

$$h_n^{(2)}(k_0 r') \approx j^{n+1} \frac{e^{-jk_0 r'}}{k_0 r'} \qquad (3.25)$$

A simpler approach will be used. It is based on the fact that, when the source of radiation is distant, the field incident on the ensemble of bodies is a plane wave.

Consider a point source at the remote point of position vector $\overline{r}' \equiv (r', \theta', \phi')$ of vector current moment $\overline{p}$. We are interested in the field of the dipole near the origin and so consider the two polarisations $\hat{p} = \hat{\theta}'$ or $\hat{\phi}'$. (The third, $\hat{r}'$, produces no field near the origin.) The incident electric field at the origin is

$$\overline{E}^i(\overline{r} = 0) = \hat{p} E_0 \qquad (3.26)$$

where

$$E_0 = -pjk_0 Z_0 \frac{e^{-jk_0 r'}}{4\pi k_0 r'} \qquad (3.27)$$

is the scalar field value at the origin.

The field in the vicinity of the origin is essentially a linearly polarised plane wave. Such a wave, with the vector value given in (3.26), has electric field vector at the point $\overline{r} \equiv (r, \theta, \phi)$ in the vicinity of the origin,

$$\overline{E}^i(\overline{r}) = \hat{p} E_0 e^{jk_0 r(\sin\theta \sin\theta' \cos(\phi - \phi') + \cos\theta \cos\theta')} \qquad (3.28)$$

We consider the two polarisations

$$\begin{aligned} \hat{p} = \hat{\theta}' = {} & \hat{r}(\sin\theta \cos\theta' \cos(\phi - \phi') - \cos\theta \sin\theta') \\ & + \hat{\theta}(\cos\theta \cos\theta' \cos(\phi - \phi') + \sin\theta \sin\theta') - \hat{\phi} \cos\theta' \sin(\phi - \phi') \end{aligned} \qquad (3.29)$$

and

$$\hat{p} = \hat{\phi}' = -\hat{r} \sin\theta \sin(\phi - \phi') + \hat{\theta} \cos\theta \sin(\phi - \phi') + \hat{\phi} \cos(\phi - \phi'). \qquad (3.30)$$

Through application of the formulas in Appendix A, (3.28) to (3.30) may be manipulated into the form

$$\overline{E}^i(\bar{r}) = E_0 \sum_{m=-\infty}^{m=\infty} \bar{c}_m(\bar{r},\theta')e^{jm(\phi-\phi')} \tag{3.31}$$

where, for $\hat{p} = \hat{\theta}'$,

$$\begin{aligned}\bar{c}_m(\bar{r},\theta') = \bar{c}_m^{\theta}(\bar{r},\theta') = j^m(-\hat{r}(j\sin\theta\cos\theta'\,\dot{J}_m(k_0 r\sin\theta\sin\theta') + \cos\theta\sin\theta'\,J_m)\\ -\hat{\theta}(j\cos\theta\cos\theta'\,\dot{J}_m - \sin\theta\sin\theta'\,J_m)\\ +\hat{\phi}\cos\theta'\frac{m}{k_0 r\sin\theta\sin\theta'}J_m)e^{jk_0 r\cos\theta\cos\theta'}\end{aligned} \tag{3.32}$$

and for $\hat{p} = \hat{\phi}'$,

$$\begin{aligned}\bar{c}_m(\bar{r},\theta') = \bar{c}_m^{\phi}(\bar{r},\theta') = j^m(\frac{m}{k_0 r\sin\theta\sin\theta'}(\hat{r}\sin\theta J_m(k_0 r\sin\theta\sin\theta')\\ -\hat{\theta}\cos\theta J_m) - \hat{\phi}j\dot{J}_m)e^{jk_0 r\cos\theta\cos\theta'}\end{aligned} \tag{3.33}$$

in which J_m is the Bessel function of order m, $\dot{J}_m$ is its derivative with respect to the argument, and the argument in all occurrences is $k_0 r\sin\theta\sin\theta'$, as shown explicitly in the first.

A free-space dyadic Green's function for a distant source is extracted from these results in the following fashion:

With the basic property of the Green's function, Cf. (3.1),

$$\overline{E}^i(\bar{r}) = \overline{\overline{G}}^0(\bar{r},\bar{r}').\bar{p} \quad . \tag{3.34}$$

(3.27) and (3.31) to (3.33) lead to

$$\overline{\overline{G}}^0(\bar{r},\bar{r}')_{r'\to\infty} = \sum_{m=-\infty}^{\infty} \overline{\overline{G}}_m^0(\bar{r},\bar{r}')_{r'\to\infty} e^{jm(\phi-\phi')} \tag{3.35}$$

where

$$\overline{\overline{G}}_m^0(\bar{r},\bar{r}')_{r'\to\infty} = -jk_0 Z_0 \frac{e^{-jk_0 r'}}{4\pi r'}(\bar{c}_m^{\theta'}(\bar{r},\theta')\hat{\theta}' + \bar{c}_m^{\phi'}(\bar{r},\theta')\hat{\phi}') \tag{3.36}$$

It will sometimes be convenient to express this in the form, (note (3.27)),

$$\overline{\overline{G}}_m^0(\bar{r},\bar{r}')_{r'\to\infty} = \frac{E_0}{p}(\bar{c}_m^{\theta'}(\bar{r},\theta')\hat{\theta}' + \bar{c}_m^{\phi'}(\bar{r},\theta')\hat{\phi}') \quad . \tag{3.37a}$$

This equation may be expressed alternatively in the following notation:

$$\overline{\overline{G}}{}_m^0(\bar{r},\bar{r}')_{r'\to\infty} = \frac{E_0}{p} \sum_{\alpha,\beta'} \hat{\alpha}\hat{\beta}'\, c_{\alpha;m}^{\beta'}(\bar{r},\theta'), \qquad \alpha = r,\theta,\phi;\ \beta' = \theta',\phi' \tag{3.37b}$$

where $c_{\alpha;m}^{\beta'}$ is the component of $\bar{c}_m^{\beta'}$ in the α direction.

3.4 Remote field point

This is the reciprocal of the previous case. It is readily found that

$$\overline{\overline{G}}^0(\bar{r},\bar{r}')_{r\to\infty} = \sum_{m=-\infty}^{\infty} \overline{\overline{G}}_m^0(\bar{r},\bar{r}')_{r\to\infty} e^{jm(\phi-\phi')} \quad , \tag{3.38}$$

where

$$\begin{aligned}\overline{\overline{G}}_m^0(\bar{r},\bar{r}')_{r\to\infty} &= -jk_0 Z_0 \frac{e^{-jk_0 r}}{4\pi r}(\hat{\theta}\bar{c}_m^{\theta}(\bar{r}',\theta)+\hat{\phi}\bar{c}_m^{\phi}(\bar{r}',\theta)) \\ &= -jk_0 Z_0 \frac{e^{-jk_0 r}}{4\pi r}\sum_{\alpha,\beta'} \hat{\alpha}\hat{\beta}' c_{\beta';m}^{\alpha}(\bar{r}',\theta), \quad \alpha=\theta,\phi;\ \beta'=r',\theta',\phi'\end{aligned} \quad . \tag{3.39}$$

3.5 Green's function harmonics and rotation

Consider the free-space dyadic Green's function given in (3.23) and (3.24) and note that any field calculated from it, see (3.1), involves vectors resulting from the operation $\overline{\overline{g}}_{mn}.\overline{p}$. Expanding this in terms of its components gives

$$\overline{\overline{g}}_{mn}\cdot\overline{p} = (g_{rr'}p_{r'} + g_{r\theta'}p_{\theta'} + g_{r\phi'}p_{\phi'})\hat{r} + (g_{\theta r'}p_{r'} + g_{\theta\theta'}p_{\theta'} + g_{\theta\phi'}p_{\phi'})\hat{\theta}$$
$$+ (g_{\phi r'}p_{r'} + g_{\phi\theta'}p_{\theta'} + g_{\phi\phi'}p_{\phi'})\hat{\phi} \tag{3.40}$$

in which $p_{r'}, p_{\theta'}, p_{\phi'}$ are respectively the r', θ', ϕ' components of $\overline{p}$ at its location $\overline{r}'$.

It is seen from (3.22) that the scalar components of the vector in the right side of (3.40) are not functions of ϕ. From the remark preceding (2.7) it follows that the vector $\overline{\overline{g}}_{mn}.\overline{p}$ transforms between points related by $\overline{r}_2 = \overline{\overline{R}}(\Delta\phi).\overline{r}_1$ as in (2.7), i.e.,

$$\overline{\overline{g}}_{mn}(\overline{r}_2,\overline{r}').\overline{p} = \overline{\overline{R}}(\Delta\phi).\overline{\overline{g}}_{mn}(\overline{r}_1,\overline{r}').\overline{p} \quad . \tag{3.41}$$

From this, noting that $\overline{p}$ is arbitrary, we have the dyadic identity

$$\overline{\overline{g}}_{mn}(\overline{r}_2,\overline{r}') = \overline{\overline{R}}(\Delta\phi).\overline{\overline{g}}_{mn}(\overline{r}_1,\overline{r}') \quad , \tag{3.42}$$

and hence, noting (3.24),

$$\overline{\overline{G}}{}^0_m(\overline{\overline{R}}(\Delta\phi).\overline{r},\overline{r}') = \overline{\overline{R}}(\Delta\phi).\overline{\overline{G}}{}^0_m(\overline{r},\overline{r}') \quad . \tag{3.43}$$

This property will be shared by any dyadic whose scalar components, expressed in spherical coordinates, are independent of ϕ.

Similarly, because the elements of $\overline{\overline{G}}{}^0_m(\overline{r},\overline{r}')$ are independent of ϕ',

$$\overline{\overline{G}}{}^0_m(\overline{r},\overline{\overline{R}}(\Delta\phi).\overline{r}') = \overline{\overline{G}}{}^0_m(\overline{r},\overline{r}').\overline{\overline{R}}{}^{-1}(\Delta\phi) \quad . \tag{3.44}$$

where $\overline{\overline{R}}{}^{-1}(\Delta\phi)$ is the dyadic which reverses the rotation $\overline{\overline{R}}(\Delta\phi)$.

Finally, it is convenient here to note the transform property of the free-space Green's function

$$\overline{\overline{G}}{}^0(\overline{\overline{R}}(\Delta\phi).\overline{r},\overline{\overline{R}}(\Delta\phi).\overline{r}') = \overline{\overline{R}}(\Delta\phi).\overline{\overline{G}}{}^0(\overline{r},\overline{r}').\overline{\overline{R}}{}^{-1}(\Delta\phi). \tag{3.45}$$

4. The incident electric field and the induced currents

We now consider the field of the point dipole to be incident on the ensemble of bodies described in Section 2. To distinguish it as such it is superscripted i.

(3.1) and (3.23) give

$$\bar{E}^i = \sum_{m=-\infty}^{\infty} \bar{E}^i_m(\bar{r}) \qquad (4.1)$$

where

$$\bar{E}^i_m(\bar{r}) = \bar{\bar{G}}^0_m(\bar{r}.\bar{r}').\bar{p}e^{jm(\phi-\phi')} \qquad (4.2)$$

and $\bar{\bar{G}}^0_m(\bar{r},\bar{r}')$ is given by (3.24) or (3.36) as appropriate.

(4.2) and (3.43) show that $\bar{E}^i_m(\bar{r})$ varies between corresponding points $\bar{r}_0$ and $\bar{r}_i$ as

$$\bar{E}^i_m(\bar{r}_i) = \bar{\bar{R}}_i.\bar{E}^i_m(\bar{r}_0)e^{jm\Phi_i} \qquad . \qquad (4.3)$$

in which Φ_i, defined in (2.1), is the azimuthal angle between the points.

(4.1) and (4.2) represent the field as the sum of a series of harmonics of the azimuthal coordinate ϕ. A given harmonic, the mth, say, takes vector values which differ at the corresponding points $\bar{r}_i$ and $\bar{r}_0$ on the bodies S_i and S_0 by rotation through angle Φ_i about the z axis, and phase shift $m\Phi_i$. The rotation of the field vector is the same as that of the bodies themselves, whence, the field experienced by each body relative to its own orientation is the same except for a phase shift.

It is concluded that the vector current distributions induced by the mth field harmonic are the same on all bodies, except for rotation and phase shift, thus:

$$\bar{J}_{im}(\bar{r}_i) = \bar{\bar{R}}_i.\bar{J}_{0m}(\bar{r}_0)e^{jm\Phi_i} \qquad (4.4)$$

where $\bar{J}_{im}(\bar{r}_i)$ is the surface current density at $\bar{r}_i$ on S_i induced by the mth field harmonic $\bar{E}^i_m$.

As the bodies are perfectly conducting, the currents flow on their surfaces. The quantities in (4.4) are surface current densities, dimensioned A/m. The conventional symbol for surface current density is $\bar{J}^S$, but in this work the superscript s will be suppressed.

(4.4) reveals a major benefit of the harmonic representation of the field: it will be necessary to determine the current distributions on only one of the bodies, since all are similar except for orientation and phase shift.

5. The electric field integral equation

Each current harmonic induced on the surfaces radiates a part of the scattered field, $\overline{E}^s_m(\bar{r})$, say. For the present we shall refer to this as the mth "harmonic" of the scattered field, although this will later be seen to be a misnomer. The mth harmonic of the total field is $\overline{E}^s_m(\bar{r})$ plus the mth harmonic of the incident field, $\overline{E}^i_m(\bar{r})$. The total mth harmonic satisfies the boundary condition that everywhere on the surfaces $S_0,\ldots,S_{I-1}$ its tangential component is zero. This is expressed in an integral equation whose solution is the mth surface current density distribution.

The field $\overline{E}^s_m(\bar{r})$ is produced by the currents on all the bodies, $\bar{J}_{im}(\bar{r}_i)$, $i = 0,\ldots,I-1$. It is given in terms of the dyadic Green's function (Cf. (3.6)) by the sum

$$\overline{E}^s_m(\bar{r}) = \sum_{i=0}^{I-1} \iint_{\bar{r}_i' \text{ on } S_i} \overline{\overline{G}}{}^0(\bar{r},\bar{r}_i').\bar{J}_{im}(\bar{r}_i')ds_i' \,. \tag{5.1}$$

Where ds_i' is an element of area at $\bar{r}_i'$ on S_i.

With (2.4) and (4.4) this is rewritten in the form

$$\overline{E}^s_m(\bar{r}) = \sum_{i=0}^{I-1} e^{jm\Phi_i} \iint_{\bar{r}_0' \text{ on } S_0} \overline{\overline{G}}{}^0(\bar{r},\overline{\overline{R}}_i.\bar{r}_0').\overline{\overline{R}}_i.\bar{J}_{0m}(\bar{r}_0')ds_0' \tag{5.2}$$

The boundary condition is

$$[\overline{E}^i_m(\bar{r}_j) + \overline{E}^s_m(\bar{r}_j)]_t = 0\,, \quad \text{all } \bar{r}_j \text{ on } S_j,\ j = 0,1,\ldots,I-1 \tag{5.3}$$

where the notation $[.]_t$ specifies the vector component tangential to the surface.

With (4.2) and (5.2), (5.3) becomes

$$\Big[\sum_{i=0}^{I-1} e^{jm\Phi_i} \iint_{\bar{r}_0' \text{ on } S_0} \overline{\overline{G}}{}^0(\bar{r}_j,\overline{\overline{R}}_i.\bar{r}_0').\overline{\overline{R}}_i.\bar{J}_{0m}(\overline{\overline{R}}_i.\bar{r}_0')ds_0' \Big]_t$$

$$= -[\overline{\overline{G}}{}^0_m(\bar{r}_j,\bar{r}').\bar{p}]_t\, e^{jm(\phi_j - \phi')}, \tag{5.4}$$

$$\text{all } \bar{r}_j \text{ on } S_j, \text{ all } j.$$

In the right side of this equation $\overline{\overline{G}}{}^0_m(\bar{r}_j,\bar{r}')$ is determined from (3.24) or (3.36) as appropriate.

(5.4) must be satisfied for all j. Explicitly, for j = 0,

$$\Big[\sum_{i=0}^{I-1} e^{jm\Phi_i} \iint_{\bar{r}_0' \text{ on } S_0} \bar{\bar{G}}^0(\bar{r}_0, \bar{\bar{R}}_i.\bar{r}_0').\bar{\bar{R}}_i.\bar{J}_{0m}(\bar{r}_0')ds_0'\Big]_t = -[\bar{\bar{G}}^0_m(\bar{r}_0,\bar{r}').\bar{p}]_t e^{jm(\phi_0-\phi')}, \text{ all } \bar{r}_0 \text{ on } S_0 \tag{5.5}$$

With (2.4) and (4.4), this may be written, after some rearrangement

$$\Big[\sum_{i=0}^{I-1} e^{jm\Phi_{i+j}} \iint_{\bar{r}_0' \text{ on } S_0} \bar{\bar{R}}_j.\bar{\bar{G}}^0(\bar{r}_0,\bar{r}_i').\bar{\bar{R}}_j^{-1}.\bar{\bar{R}}_{i+j}.\bar{J}_{0m}(\bar{r}_0')ds_0'\Big]_t = -[\bar{\bar{G}}^0_m(\bar{r}_j,\bar{r}').\bar{p}]_t e^{jm(\phi_j-\phi')}]_t, \ \bar{r}_j \text{ on } S_j \tag{5.6}$$

in which it has been noted that $\Phi_i + \Phi_j = \Phi_{i+j}$ and $\bar{\bar{R}}_i = \bar{\bar{R}}_j^{-1}.\bar{\bar{R}}_{i+j}$.

Using, see (3.45),

$$\bar{\bar{R}}_j.\bar{\bar{G}}^0(\bar{r}_0,\bar{r}_i').\bar{\bar{R}}_j^{-1} = \bar{\bar{G}}^0(\bar{r}_j,\bar{r}_{i+j}') \tag{5.7}$$

and replacing the summation index i by i+j, we may recover (5.4) for $\bar{r}_j$ on S_j. Thus satisfying the condition on S_0 satisfies it on all.

In (5.5), multiply both sides by $e^{jm\phi'}$ and change the order of summation and integration to get

$$\Big[\iint_{\bar{r}_0' \text{ on } S_0} \bar{\bar{G}}^{\Sigma}_m(\bar{r}_0,\bar{r}_0').\bar{J}_m(\bar{r}_0')ds_0'\Big]_t = -[\bar{\bar{G}}^0_m(\bar{r}_0,\bar{r}').\bar{p}]_t e^{jm\phi_0}, \text{ all } \bar{r}_0 \text{ on } S_0 \tag{5.8}$$

where

$$\bar{\bar{G}}^{\Sigma}_m(\bar{r}_0,\bar{r}_0') = \sum_{i=0}^{I-1} e^{jm\Phi_i}\bar{\bar{G}}^0(\bar{r}_0,\bar{\bar{R}}_i.\bar{r}_0').\bar{\bar{R}}_i \quad , \tag{5.9}$$

and

$$\bar{J}_m(\bar{r}_0') = \bar{J}_{0m}(\bar{r}_0')e^{jm\phi'} \quad . \tag{5.10}$$

The azimuthal position of the source ϕ' does not appear in (5.8) and hence the new unknown $\bar{J}_m(\bar{r}_0')$ is independent of ϕ'. This is a further benefit of the present approach: if we are interested, as we shall be, in source positions with a range of azimuthal angles ϕ', all with the same polar angle θ', it is unnecessary to solve more than one integral equation, (5.8). The currents on all bodies, for all azimuthal incidence directions are then, from (4.4) and (5.10),

$$\bar{J}_{im}(\bar{r}_i) = \bar{\bar{R}}_i . \bar{J}_m(\bar{r}_0) e^{jm(\Phi_i - \phi')} \quad . \tag{5.11}$$

A case of special interest is that in which the source point is distant. Let $\bar{J}_m^{\beta'}(\bar{r}_0')_{r' \to \infty}$, $\beta' = \theta', \phi'$ be the current distribution when the source has polarisation $\bar{p} = \hat{\beta}' p$. Then with (3.37a) and (5.8), $\bar{J}_m^{\beta'}(\bar{r}_0')_{r' \to \infty}$ is the solution of

$$\Big[\iint_{\bar{r}_0' \,\mathrm{on}\, S_0} \bar{\bar{G}}^{\Sigma}(\bar{r}_0, \bar{r}_0') . \bar{J}_m^{\beta'}(\bar{r}_0')_{r' \to \infty} ds_0' \Big]_t = -E_0 [\bar{c}_m^{\beta'}(\bar{r}_0, \theta')]_t e^{jm\phi_0} , \tag{5.12}$$

$$\text{all } \bar{r}_0 \text{ on } S_0$$

6. The scattered field

The scattered field is the sum of the fields radiated by all the harmonic currents on all the bodies. It is given by the formula

$$\overline{\overline{E}}^{S}(\bar{r}) = \sum_{\ell=-\infty}^{\infty} \sum_{i=0}^{I-1} \iint_{\bar{r}_i' \text{ on } S_i} \overline{\overline{G}}^{0}(\bar{r},\bar{r}_i').\bar{J}_{i\ell}(\bar{r}_i')ds_i'. \tag{6.1}$$

Substitution using (3.23) and (5.11) and slight rearrangement leads to

$$\overline{\overline{E}}^{S}(\bar{r}) = \sum_{\ell=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} e^{j(m\phi+\ell\phi')} \sum_{i=0}^{I-1} e^{-j(\ell+m)\Phi_i} \iint_{\bar{r}_0' \text{ on } S_0} \overline{\overline{G}}_m^{0}(\bar{r},\bar{r}_0').\bar{J}_{\ell}(\bar{r}_0')e^{-jm\phi'_0}ds_0' \tag{6.2}$$

Finally we note from the definition of Φ_i, see (2.1), that the summation on i is zero unless $\ell+m$ is an integer multiple of I, in which case it equals I. Therefore set $\ell = nI - m$ and sum over integer values of n to get

$$\overline{\overline{E}}^{S}(\bar{r}) = I \sum_{m=-\infty}^{\infty} e^{jm(\phi-\phi')} \sum_{n=-\infty}^{\infty} e^{jnI\phi'} \iint_{\bar{r}_0' \text{ on } S_0} \overline{\overline{G}}_m^{0}(\bar{r},\bar{r}_0').\bar{J}_{nI-m}(\bar{r}_0')e^{-jm\phi_0'}ds_0' \quad (6.3$$

7. The Green's function for the stationary ensemble

7.1 The general case

In (6.3), $\overline{E}^S(\bar{r})$ is the scattered field at $\bar{r}$ when the point dipole $\overline{p}$ at $\bar{r}'$ radiates in the presence of the ensemble of bodies. This may be expressed in terms of a dyadic Green's function relating the scattered field to the dipole thus (Cf. (3.1)):

$$\frac{1}{p}\overline{E}^S(\bar{r}) = \overline{\overline{G}}^S(\bar{r},\bar{r}').\hat{p} \qquad . \tag{7.1}$$

$\overline{\overline{G}}^s(\bar{r},\bar{r}')$ (note the lower-case superscript) may be expanded in terms of its component dyads in spherical coordinates as follows:

$$\overline{\overline{G}}^s(\bar{r},\bar{r}') = \sum_{\alpha,\beta'} G^s_{\alpha\beta'}(\bar{r},\bar{r}')\hat{\alpha}\hat{\beta}' \qquad \alpha = r,\theta,\phi;\ \beta' = r',\theta',\phi' \tag{7.2}$$

Let the dipole be oriented in the β' direction, i.e. $\overline{p} = \hat{\beta}'p$; for this case let $\bar{J}^{\beta'}_m(\bar{r}_0{}')$ be the solution of (5.8), $\overline{E}^{s\beta'}$ be the scattered field calculated from (6.3) and $E^{s\beta'}_\alpha$ be its α-component. It is seen from (7.1) and (7.2) that

$$G^s_{\alpha\beta'} = \frac{1}{p}E^{s\beta'}_\alpha(\bar{r}) \qquad .$$

With this result and (6.3) $\overline{\overline{G}}^s(\bar{r},\bar{r}')$ may be written as a double summation of terms $\overline{\overline{G}}^s_{mn}(\bar{r},\bar{r}')$:

$$\begin{aligned}\overline{\overline{G}}^s(\bar{r},\bar{r}') &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{jnI^S\phi'}\overline{\overline{G}}^s_{mn}(\bar{r},\bar{r}')e^{jm(\phi-\phi')} \\ &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{jnI^S\phi'}\sum_{\alpha,\beta'}\hat{\alpha}\hat{\beta}'G^s_{\alpha\beta';mn}(\bar{r},\bar{r}')e^{jm(\phi-\phi')}, \\ &\qquad \alpha = r,\theta,\phi;\ \beta' = r',\theta',\phi\end{aligned} \tag{7.3}$$

where $G^S_{\alpha\beta';mn}$ is the $\alpha\beta'$ element of $\overline{\overline{G}}^S_{mn}(\bar{r},\bar{r}')$, and

$$G^{s}_{\alpha\beta';mn}(\bar{r},\bar{r}') = \frac{1}{p} I^{S} \iint_{\bar{r}_0' \text{ on } S_0^S}$$

$$\sum_{\beta_0'} G^{0}_{\alpha\beta_0';m}(\bar{r},\bar{r}_0') J^{\beta'}_{\beta_0';nI^S-m}(\bar{r}_0') e^{-jm\phi'_0} ds_0' \quad \beta_0' = r_0', \theta_0', \phi_0'$$

(7.4a)

in which, in anticipation of later developments, I has been replaced by I^S and S_0 by S_0^S.

If the source point is distant, (7.4a) should be evaluated with the current density in the integrand given by the solution of (5.12).

If the observation point is distant, (7.4a) becomes, note (3.39),

$$G^{s}_{\alpha\beta';mn}(\bar{r},\bar{r}')_{r\to\infty} = -jk_0 Z_0 \frac{e^{-jk_0 r}}{4\pi r} \frac{1}{p} I^{S} \iint_{\bar{r}_0' \text{ on } S_0^S}$$

$$\sum_{\beta_0' = r_0', \theta_0', \phi_0'} c^{\alpha}_{\beta_0';m}(\bar{r}_0', \theta) J^{\beta'}_{\beta_0';nI^S-m}(\bar{r}_0') e^{-jm\phi'_0} ds_0'$$

(7.4b)

The total field produced by the dipole $\bar{p}$ in the presence of the ensemble is the sum of the scattered and incident fields, see figure 7.1. The dyadic Green's function for the total field is the sum of that developed in Section 3 for the free-space case and that in (7.3). Using for the former the representation in (3.23) and for the latter (7.3) we have

$$\bar{\bar{G}}^{S}(\bar{r},\bar{r}') = \sum_{m=-\infty}^{\infty} e^{jm(\phi-\phi')} \sum_{n=-\infty}^{\infty} e^{jnI^S\phi'} \bar{\bar{G}}^{S}_{mn}(\bar{r},\bar{r}') \qquad (7.5)$$

where $\bar{\bar{G}}^{S}(\bar{r},\bar{r}')$ (note the upper-case superscript) is a dyadic Green's function relating the total field to the dipole radiating in the presence of the ensemble S, and

$$\bar{\bar{G}}^{S}_{m0}(\bar{r},\bar{r}') = \delta_{n0}\bar{\bar{G}}^{0}_{m}(\bar{r},\bar{r}') + \bar{\bar{G}}^{s}_{m0}(\bar{r},\bar{r}') \qquad (7.6)$$

where δ_{ij} is the Kronecker delta, equal to one if $i = j$, otherwise zero.

(7.5) and (7.6), with the definition of their terms in (3.23) and (7.4), constitute a formal solution to the problem of the dyadic Green's function for the ensemble of bodies.

7.2 The dyadic Green's functions and rotation

In (7.4) the $G^{S}_{mn,\alpha\beta'}(\bar{r},\bar{r}')$ are not functions of ϕ or ϕ'. It follows that $\overline{\overline{G}}^{S}_{mn}(\bar{r},\bar{r}')$ transforms as in (3.43) and (3.44); specifically,

$$\begin{aligned}\overline{\overline{G}}^{S}_{mn}(\overline{\overline{R}}(\Delta\phi).\bar{r},\bar{r}') &= \overline{\overline{R}}(\Delta\phi).\overline{\overline{G}}^{S}_{mn}(\bar{r},\bar{r}'),\\ \overline{\overline{G}}^{S}_{mn}(\bar{r},\overline{\overline{R}}(\Delta\phi).\bar{r}') &= \overline{\overline{G}}^{S}_{mn}(\bar{r},\bar{r}').\overline{\overline{R}}^{-1}(\Delta\phi)\end{aligned} \tag{7.7}$$

It has been noted in (3.43) and (3.44) that $\overline{\overline{G}}^{0}_{m}(\bar{r},\bar{r}')$ transforms in the same fashion. It follows from (7.6) that $\overline{\overline{G}}^{S}_{mn}(\bar{r},\bar{r}')$ also transforms in this way:

$$\begin{aligned}\overline{\overline{G}}^{S}_{mn}(\overline{\overline{R}}(\Delta\phi).\bar{r},\bar{r}') &= \overline{\overline{R}}(\Delta\phi).\overline{\overline{G}}^{S}_{mn}(\bar{r},\bar{r}'),\\ \overline{\overline{G}}^{S}_{mn}(\bar{r},\overline{\overline{R}}(\Delta\phi).\bar{r}') &= \overline{\overline{G}}^{S}_{mn}(\bar{r},\bar{r}').\overline{\overline{R}}^{-1}(\Delta\phi)\end{aligned} \tag{7.8}$$

These two formulas together give

$$\overline{\overline{G}}^{S}_{mn}(\overline{\overline{R}}(\Delta\phi).\bar{r},\overline{\overline{R}}(\Delta\phi).\bar{r}') = \overline{\overline{R}}(\Delta\phi).\overline{\overline{G}}^{S}_{mn}(\bar{r},\bar{r}').\overline{\overline{R}}^{-1}(\Delta\phi) \tag{7.9}$$

By reason of its dependence on ϕ and ϕ', $\overline{\overline{G}}^{S}(\bar{r},\bar{r}')$ transforms in a slightly more complicated fashion:

$$\begin{aligned}\overline{\overline{G}}^{S}(\overline{\overline{R}}(\Delta\phi).\bar{r},\bar{r}') &= \sum_{m=-\infty}^{\infty} e^{jm(\phi-\phi')} \sum_{n=-\infty}^{\infty} e^{jnI^{S}\phi'}\overline{\overline{R}}(\Delta\phi).\overline{\overline{G}}^{S}_{mn}(\bar{r},\bar{r}')e^{jm\Delta\phi}\\ \overline{\overline{G}}^{S}(\bar{r},\overline{\overline{R}}(\Delta\phi).\bar{r}') &= \sum_{m=-\infty}^{\infty} e^{jm(\phi-\phi')} \sum_{n=-\infty}^{\infty} e^{jnI^{S}\phi'}\overline{\overline{G}}^{S}_{mn}(\bar{r},\bar{r}').\overline{\overline{R}}^{-1}(\Delta\phi)\\ &\qquad e^{j(nI^{S}-m)\Delta\phi}\end{aligned} \tag{7.10}$$

and

$$\begin{aligned}\overline{\overline{G}}^{S}(\overline{\overline{R}}(\Delta\phi).\bar{r},\overline{\overline{R}}(\Delta\phi).\bar{r}') = &\sum_{m=-\infty}^{\infty} e^{jm(\phi-\phi')}\\ &\sum_{n=-\infty}^{\infty} e^{jnI^{S}\phi'}\overline{\overline{R}}(\Delta\phi).\overline{\overline{G}}^{S}_{mn}(\bar{r},\bar{r}').\overline{\overline{R}}^{-1}(\Delta\phi)e^{jnI^{S}\Delta\phi}\end{aligned} \tag{7.11}$$

7.3 The dyadic Green's function for remote source and field points

For $r' \to \infty$, $\overline{\overline{G}}{}^0_m(\bar{r},\bar{r}_0{}')$ is given by (3.37) and $\overline{\overline{G}}{}^S(\bar{r},\bar{r}')$ is the special case of (7.3) with the currents in (7.4a) corresponding to a remote source. Substitution into (7.6) gives the elements of $\overline{\overline{G}}{}^S_m(\bar{r},\bar{r}')$ to be

$$G^S_{\alpha\beta';mn}(\bar{r},\bar{r}')_{r'\to\infty} = \delta_{n0}\frac{E_0}{p}c^{\beta'}_{\alpha;m}(\bar{r},\theta')$$
$$+\frac{1}{p}I^S \iint_{\bar{r}_0{}'\,\mathrm{on}\,S^S_0} \sum_{\beta_0{}'} G^S_{\alpha\beta_0{}';m}(\bar{r},\bar{r}_0{}')J^{\beta'}_{\beta_0{}';nI^S-m}(\bar{r}_0{}')e^{-jm\phi'_0}ds_0{}'$$
$$\beta_0{}' = r_0{}',\theta_0{}',\phi_0{}' \tag{7.12}$$

where $\bar{J}^{\beta'}_m$ is the solution of (5.12).

For $r \to \infty$, substitute from (3.39) and (7.4b) into (7.6) to obtain

$$G^S_{\alpha\beta';mn}(\bar{r},\bar{r}')_{r\to\infty} = -jk_0Z_0\frac{e^{-jk_0r}}{4\pi r}(\delta_{n0}c^{\alpha}_{\beta';m}(\bar{r}',\theta)$$
$$+\frac{1}{p}I^S \iint_{\bar{r}_0{}'\,\mathrm{on}\,S_0} \sum_{\beta_0{}'} c^{\alpha}_{\beta_0{}';m}(\bar{r}_0{}',\theta)J^{\beta'}_{\beta_0{}';nI^S-m}(\bar{r}_0{}')e^{-jm\phi'_0}ds_0{}')$$
$$\beta_0{}' = r_0{}',\theta_0{}',\phi_0{}' \tag{7.14}$$

where $\bar{J}^{\beta'}_m$ is the solution of (5.8).

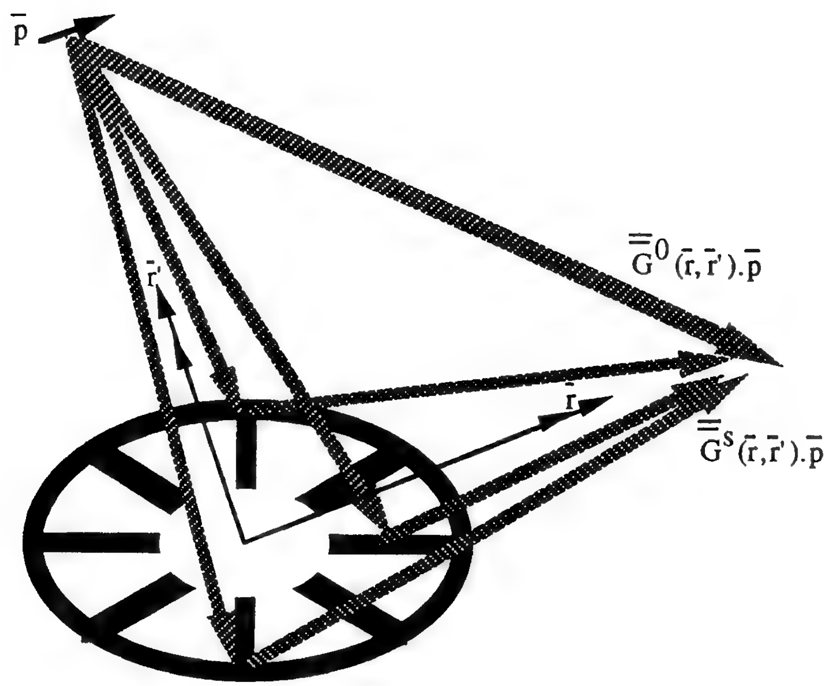

Figure 7.1: The field of the point dipole radiating in the presence of the ensemble as the sum of the free-space and scattered fields.

8. The rotating ensemble in the presence of a stationary ensemble

8.1 The ensembles

We now consider two coaxial ensembles similar to that studied in the foregoing sections, one static and the other rotating about the common axis. This is illustrated in figure 8.1.

The stationary ensemble, S^S say, consists of I^S bodies, with surfaces S_i^S, $i=0,1,...I^S-1$, the ith generated from the 0th by rotation through the angle

$$\Phi_i^S = \frac{2\pi}{I^S} i \quad . \tag{8.1}$$

The rotating ensemble, S^R say, consists of I^R bodies, with surfaces, at time t = 0, S_i^R, $i=0,1,...I^R-1$, the ith generated from the 0th by rotation through the angle

$$\Phi_i^R = \frac{2\pi}{I^R} i \quad . \tag{8.2}$$

This ensemble rotates about the z-axis in the $+\phi$ direction with angular velocity Ω.

The stationary ensemble is the environment in which we shall study scattering from the rotating ensemble. The dyadic Green's function for this environment, $\overline{\overline{G}}^S(\bar{r},\bar{r}')$ is given in (7.5).

Let a point on body S_i^R of the ensemble at reference time t = 0 be $\bar{r}_i$, with azimuthal angle ϕ_i. At time t, the ensemble has rotated through angle Ωt, and $\bar{r}_i$ and ϕ_i have become

$$\bar{r}_i^t = \overline{\overline{R}}(\Omega t).\bar{r}_i = \overline{\overline{R}}(\Phi_i + \Omega t).\bar{r}_0 \tag{8.3a}$$

and

$$\phi_i^t = \phi_i + \Omega t = \phi_0 + \Phi_i + \Omega t \tag{8.3b}$$

respectively, and $\bar{r}_i^t$ resides on a surface we shall call S_i^{Rt}. We extend the concept of corresponding points, introduced after (2.3), to include $\bar{r}_i^t$, $\bar{r}_i$ and $\bar{r}_0$ Whenever similarly sub/superscripted position vectors or their coordinates appear in the same equation they are to be understood as corresponding.

8.2 Currents on the rotating ensemble

The incident field at the point $\bar{r}_i^t$, due to a dipole of moment is $\bar{p}$ at $\bar{r}'$ acting in the presence of the stationary ensemble, is

$$\bar{E}^i(\bar{r}_i^t) = \bar{\bar{G}}^S(\bar{r}_i^t, \bar{r}').\bar{p} \tag{8.4}$$

Let the current density distribution at $\bar{r}_j^t$ on the jth body be $\bar{J}_j^t(\bar{r}_j^t)$. It is convenient to define a related function $\bar{J}_j(\bar{r}_0, \Omega t)$, through

$$\bar{J}_j^t(\bar{r}_j^t) = \bar{\bar{R}}(\Phi_j^R + \Omega t).\bar{J}_j(\bar{r}_0, \Omega t). \tag{8.5}$$

Recognising that the current on the jth body at time t is the same as that on the 0th at the time $t + \dfrac{\Phi_j^R}{\Omega}$ when it occupies the same position, we note

$$\bar{J}_j(\bar{r}_0, \Omega t) = \bar{J}_0(\bar{r}_0, \Omega t + \Phi_j^R) \tag{8.6}$$

In order that the tangential electric field on the rotating ensemble be zero, $\bar{J}_j^t(\bar{r}_j^t)$ must satisfy the integral equation (Cf.(5.4))

$$\Big[\iint_{\bar{r}_j^{t\prime} \text{ on } S_j^{Rt}} \sum_{j=0}^{I^R - 1} \bar{\bar{G}}^S(\bar{r}_i^t, \bar{r}_j^{t\prime}).\bar{J}_j^t(\bar{r}_j^{t\prime})\, ds_j^{t\prime} \Big]_t = -[\bar{\bar{G}}^S(\bar{r}_i^t, \bar{r}').\bar{p}]_t \tag{8.7}$$

$$\text{all } \bar{r}_i^t \text{ on } S_i^{Rt}, \text{ all i}$$

With substitution from (7.5), (7.7), (7.8), (8.3) and (8.5), and some rearrangement, this becomes

$$\Big[\int_{\bar{r}_0' \text{ on } S_0^R} \sum_{j=0}^{I^R - 1} \sum_{m'=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} e^{jm'(\phi_i - \phi_j')} e^{jn' I^S(\phi_j' + \Omega t)} \bar{\bar{G}}^S_{m'n'}(\bar{r}_0, \bar{r}_0').\bar{J}_j(\bar{r}_0', \Omega t)\, ds_0' \Big]_t$$

$$= -\Big[\sum_{m=-\infty}^{\infty} e^{jm(\phi_0 - \phi')} e^{jm(\Omega t + \Phi_i^R)} \bar{\bar{G}}^S_m(\bar{r}_0, \bar{r}').\bar{p} \Big]_t \tag{8.8}$$

$$\text{all } \bar{r}_0 \text{ on } S_0^R, \text{ all i}$$

where

$$\overline{\overline{G}}^{S}_{m}(\bar{r}_0,\bar{r}).\bar{p} = \sum_{n=-\infty}^{\infty} e^{jnI^S\phi'}\overline{\overline{G}}^{S}_{mn}(\bar{r}_0,\bar{r}).\bar{p} \tag{8.9}$$

The current distributions on the bodies are to be found from the solutions of the above integral equation.

Unlike the case with (5.8) there appear to be no simple relations between the $\bar{J}_j(\bar{r}_0',\Omega t)$ from one body to another or over time or between directions of incidence. Nor is it possible to reduce the surfaces over which the equation is enforced to a single one. To solve (8.8) by the Moment Method it will be necessary to distribute basis functions over all the bodies, and enforce the equation at a number of points equal to the number of bases. This must be done for a series of time values. And each direction of incidence requires a separate solution. Thus the method does not see the large reduction in matrix dimension enjoyed by the earlier equation (5.8).

The solutions $\bar{J}_j(\bar{r}_0,\Omega t)$ are periodic with fundamental frequency Ω. Their spectra are lines separated by this frequency. It is sufficient to determine only one spectrum, that of $\bar{J}_0(\bar{r}_0,\Omega t)$ say, since all the other currents are time-shifted versions of this, see (8.6). Let the number of lines to be determined be 2N+1 (one at the centre and N on each side). Then N is the integer which most closely satisfies

$$2N+1 \geq \frac{\Delta\Omega}{\Omega} \tag{8.10}$$

where $\Delta\Omega$ is the frequency bandwidth of $\bar{J}_0(\bar{r}_0,\Omega t)$, and will be discussed below.

Determination of the spectrum through use of a discrete Fourier transform requires values of $\bar{J}_0(\bar{r}_0,\Omega t)$ at 2N+1 times, evenly spaced through the rotation period $\frac{2\pi}{\Omega}$ at angular intervals $\Delta(\Omega t) = \frac{2\pi}{2N+1}$. It is noted that the solution of (8.8) for a given $\Omega t_n = n\Delta(\Omega t)$, yields $\bar{J}_j(\bar{r}_0,\Omega t_n)$ for $j = 0 \ldots I^R$, and with (8.6) we therefore have $\bar{J}_0(\bar{r}_0,(\frac{n}{2N+1}+\frac{j}{I^R})2\pi)$, $j = 0\ldots I^R$; i.e. each solution of (8.8) for a given t_n yields $\bar{J}_0(\bar{r}_0,\Omega t)$ at I^R values of Ωt. For 2N+1 spectral components we therefore need to solve the integral equation $N_{solve} = \frac{2N+1}{I^R}$ times, for $\Omega t_n = n\Delta(\Omega t)$, $n = 0,\ldots,N_{solve}-1$. This corresponds to solutions at

$$\Omega t_n = \frac{n}{N_{solve}}\frac{2\pi}{I^R},\ n = 0,\ldots,N_{solve}-1 \tag{8.11}$$

i.e. at a set of N_{solve} equi-spaced intervals in the inter-body angle.

With (8.10), N_{solve} is the integer which most closely satisfies

$$N_{solve} \geq \frac{\Delta\Omega}{\Omega I^{R}} \tag{8.12}$$

8.3 The current spectra

Assuming that (8.8) has been solved a sufficient number of times and a discrete Fourier transform to have been applied, we may write $\bar{J}_0(\bar{r}_0,\Omega t)$ as the series of frequency components

$$\bar{J}_0(\bar{r}_0,\Omega t)= \sum_{\ell=-\infty}^{\infty} \bar{J}_{0\ell}(\bar{r}_0)e^{j\ell\Omega t} \tag{8.12}$$

where $\bar{J}_{0\ell}(\bar{r}_0)$, $\ell = 0,\pm1,\pm2,...,\pm N$ is the discrete Fourier transform of $\bar{J}_0(\bar{r}_0,\Omega t_n)$ as determined above, and, with (8.5) and (8.6),

$$\bar{J}_j^t(\bar{r}_j^t) = \bar{\bar{R}}(\Phi_j^R + \Omega t). \sum_{\ell=-\infty}^{\infty} \bar{J}_{0\ell}(\bar{r}_0)e^{j\ell(\Omega t+\Phi_j^R)} \tag{8.13}$$

It was seen in [1,2] that, for a rotating ensemble in free space, the current spectrum extends in principle over an infinite band of frequencies, but becomes negligible outside a finite range. The extremes of the significant spectrum correspond to the Doppler shifts associated with the greatest linear velocities of any point on the rotating ensemble, as viewed by the source. This is greatest when the source lies in the plane of rotation, in which case the significant frequencies extend $k_0R_{max}\Omega$ (where R_{max} is the maximum radius of the ensemble) above and below the centre. The current components with the extreme frequencies, due to the rotation, radiate fields with spectra extending over similar bands, again understandable in terms of Doppler shift. The backscattered field has significant frequency components extending $2k_0R_{max}\Omega$ above and below the centre.

In the composite of the stationary and rotating ensembles, currents in each ensemble radiate fields which induce currents in the other ensemble. Thus a frequency component in one ensemble, because of its motion relative to the other, induces a spectrum of currents in the other, and vice versa. A set of current frequencies in one extending over a band $\pm k_0R_{max}\Omega$ about the centre induces a set of frequencies in the other over a band $\pm 2k_0R_{max}\Omega$, which in turn induces a band $\pm 3k_0R_{max}\Omega$ in the first, and so on. The resulting bandwidth is infinite. However each extension of the band occurs through electromagnetic interaction between the currents in the two ensembles. If the mutual coupling is weak, as it may be if the ensembles are separated by more than a small part of a wavelength, each extension is weaker than the previous by a constant ratio. The current spectrum is therefore expected to be significant over the first few ranges, two or three times $\pm k_0R_{max}\Omega$ perhaps, and to decrease exponentially outside this.

An estimate of $\Delta\Omega$ is therefore $2N_{int}k_0R_{max}\Omega$, where N_{int} is the number of interactions between the ensembles considered significant (two or three, say). Then with (8.12), and noting that the arc distance between the points on the bodies at maximum radius is $d = R_{max}\frac{2\pi}{I^R}$, we have

$$N_{solve} \geq N_{int} d_{\lambda/2}$$

where $d_{\lambda/2}$ is d measured in half-wavelengths of the illumination frequency.

The above simple relation gives the number of times (8.8) must be solved. With (8.11) we have the rotational positions at which solutions are required:

$$\Omega t_n = n \frac{1}{N_{int} d_{\lambda/2}} \frac{2\pi}{I^R}, \ n = 0, \ldots, N_{int} d_{\lambda/2} - 1$$

i.e. at each of $N_{int} d_{\lambda/2}$ equi-spaced subdivisions of the angle between the bodies.

Experience of calculating such spectra suggests that the value $N_{int} = 2$ provides good results [7].

The above formulas are essentially a reflection of the Nyquist sampling criterion. Seen in this light they therefore indicate the number and separation of determinations of scattering which are necessary whether by the present method or alternative calculation or measurement.

8.4 The scattered field

The field radiated by the rotor currents is the sum of the fields radiated by all the currents on all the bodies, in the presence of the stationary ensemble S^S. Thus,

$$\bar{E}^s(\bar{r},t) = \sum_{i=0}^{I-1} \iint_{\bar{r}_i^{t'} \text{ on } S_i^{Rt}} \bar{\bar{G}}^S(\bar{r},\bar{r}_i^{t'}).\bar{J}_i^t(\bar{r}_i^{t'})ds_i^{t'}. \tag{8.14}$$

With (7.10), and noting that the rotation from $\bar{r}_0'$ to $\bar{r}_i^{t'}$ involves the azimuthal rotation $\Delta\phi = \Phi_i^R + \Omega t$, we represent $\bar{\bar{G}}^S(\bar{r},\bar{r}_i^{t'})$ in the form

$$\bar{\bar{G}}^S(\bar{r},\bar{r}_i^{t'}) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{jm(\phi-\phi_0')}e^{jnI^S\phi_0'} e^{j(nI^S-m)(\Phi_i^R+\Omega t)}\bar{\bar{G}}^S_{mn}(\bar{r},\bar{r}_0').\bar{\bar{R}}^{-1}(\Phi_i^R+\Omega t) \tag{8.15}$$

With (8.13) and (8.15), (8.14) becomes after rearrangement

$$\bar{E}^s(\bar{r},t) = \sum_{\ell=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{j(nI^S-m+\ell)\Omega t}e^{jm\phi} \iint_{\bar{r}_0' \text{ on } S_0} e^{j(nI^S-m)\phi_0'}\bar{\bar{G}}^S_{mn}(\bar{r},\bar{r}_0').\bar{J}_{0\ell}(\bar{r}_0)ds_0' \sum_{i=0}^{I-1} e^{j(nI^S-m+\ell)\Phi_j^R} \tag{8.16}$$

From the definition of Φ_i^R, see (8.2), the summation on i is zero unless $nI^S - m + \ell$ is an integer multiple, h say, of I^R, in which case it equals I^R. Therefore, set $\ell = hI^R - nI^S + m$ and sum over h. (8.16) now takes the form

$$\bar{E}^s(\bar{r},t) = \sum_{h=-\infty}^{\infty} \bar{E}^s_{hI^R}(\bar{r})e^{j(\omega_0 + hI^R\Omega)t} \tag{8.17}$$

where

$$\bar{E}^s_{hI^R}(\bar{r}) = I^R \sum_{\ell=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{j(nI^S-hI^R+\ell)\phi} \iint_{\bar{r}_0' \text{ on } S_0^R} \sum_{n=-\infty}^{\infty} e^{j(hI^R-\ell)\phi_0'} \bar{\bar{G}}^S_{nI^S-hI^R+\ell,n}(\bar{r},\bar{r}_0').\bar{J}_{0,\ell}(\bar{r}_0'))ds_0' \tag{8.18}$$

and the hitherto suppressed time dependence $e^{j\omega_0 t}$ has been reinserted.

8.5 The spectrum of the scattered field

(8.17) represents the field as a spectrum of components, with radian frequencies

$$\omega_{hI^R} = \omega_0 + hI^R\Omega,\ h = 0, \pm1, \pm2, \ldots, \pm\infty \tag{8.19}$$

This is consistent with expectations. In time $\frac{2\pi}{I^R\Omega}$ each body rotates to the position previously occupied by its nearest neighbour and the ensemble represents the same scattering geometry. Thus the scattered field is modulated periodically with the fundamental radian frequency $I^R\Omega$ and the spectrum consists of lines separated by this frequency.

An insight drawn from the earlier studies [1,2] is that the values of indices $nI^S - hI^R + \ell$ and n of $\overline{\overline{G}}^S_{nI^S - hI^R + \ell, n}(\bar{r}, \bar{r}_0')$ are limited to finite ranges. However it has been seen in Section 8.3 above that the current index ℓ may have an extended range due to multiple interactions between the stationary and rotating ensembles. It follows that hI^R may have a correspondingly extended range.

Thus the spectrum of the rotating ensemble in the presence of the stationary bodies extends beyond the limits observed in the case of the ensemble rotating in free space. As observed for the induced currents, this may be attributed to the electromagnetic interaction between the two ensembles, and will be greater or less as the interaction is stronger or weaker. It may be understood as a Doppler effect due to the changing distances between the interacting bodies.

A clear example of the extended spectrum is seen in the case of axial backscatter. For the ensemble rotating in free space there are no frequency components other than the frequency of illumination. In the presence of the stationary ensemble, the spectrum has been found by independent calculations to extend over a range of components[7].

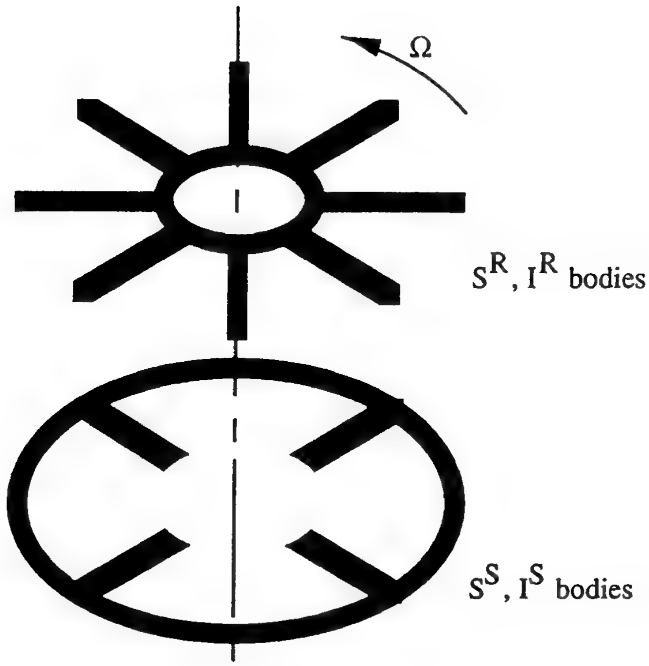

Figure 8.1: Two coaxial ensembles, the upper rotating, the lower stationary.

9. Conclusions

The study of electromagnetic scatter from rotating objects with angular periodicity has proceeded through three stages: (i) modelling of the objects as wires radiating from an axis, reported in [1]; (ii) generalisation of the wires to include bodies of arbitrary shape, reported in [2]; and (iii) the inclusion of an ensemble of stationary bodies coaxial with the rotating ensemble, the subject of the present report.

A procedure has been developed by which the spectrum of the scattered field may be calculated. As has been the case previously, it is possible to deduce some of the characteristics of the spectrum from the mathematical form of the equations, without proceeding to numerical solution. The main conclusions of this nature are that the spectral lines are spaced at intervals of the "body rate", i.e. the rotation frequency multiplied by the number of bodies in the rotating ensemble, and that the spectrum extends beyond the limits of the Doppler shift associated with the linear velocity of the bodies. These conclusions are supported by independent calculations [7].

A question which is not answered here concerns the possibility of specially strong spectral components associated with simple combinations of the numbers of bodies in the two ensembles, e.g. $(mI^R + nI^S)\Omega$, where m and n are small integers. The appearance of the expressions such as $nI^S - hI^R$ in (8.18) is intriguing, but no conclusions have so far come to light. If such characteristics exist, it may be necessary to seek them in numerical solutions.

It would be possible to extend these studies to include further ensembles of bodies, stationary and rotating, with different numbers of bodies. Indeed, in so far as the present work includes the extreme case where there is only one body in each ensemble, it includes multiple layers of stationary and synchronously rotating ensembles, since all the stationary ensembles may be viewed as a single ensemble of one body, and the same is true of the synchronously rotating ensembles.

However, as the models which are the subjects of this series of studies have become progressively more general and complicated, the opportunities for exploiting the angular periodicities have reduced. The analysis has become progressively more laborious and the properties of the spectrum more difficult to state categorically.

For these reasons it seems probable that further progress will best be made through numerical and experimental studies. The proposed continuation of the work by the present author lies in the first of these directions.

Appendix A

The following formulas may be found in, or readily derived from formulas found in, [6, Ch 9]:

$$e^{j\rho\cos\phi} = \sum_{n=-\infty}^{\infty} j^n J_n(\rho) e^{jn\phi} \tag{A1}$$

$$\begin{aligned} \cos\phi e^{j\rho\cos\phi} &= \sum_{n=-\infty}^{\infty} j^{n-1} J_n'(\rho) e^{jn\phi} \\ &= \frac{1}{2} \sum_{n=-\infty}^{\infty} j^{n-1} (J_{n-1}(\rho) - J_{n+1}(\rho)) e^{jn\phi} \end{aligned} \tag{A2}$$

(where a prime denotes differentiation with respect to the argument)

$$\begin{aligned} \sin\phi e^{j\rho\cos\phi} &= -\sum_{n=-\infty}^{\infty} j^n \frac{n}{\rho} J_n(\rho) e^{jn\phi} \\ &= -\frac{1}{2} \sum_{n=-\infty}^{\infty} j^n (J_{n-1}(\rho) + J_{n+1}(\rho)) e^{jn\phi} \end{aligned} \tag{A3}$$

$$J_n(\rho) = \frac{\rho}{2n}(J_{n-1}(\rho) + J_{n+1}(\rho)) \tag{A4}$$

$$J_n'(\rho) = \frac{1}{2}(J_{n-1}(\rho) - J_{n+1}(\rho)) \tag{A5}$$

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19. ABSTRACT

A rotating ensemble of bodies with angular periodicity rotates in the presence of a similar, stationary ensemble of generally different periodicity. The structure is illuminated from an electromagnetic source. The scattered field is modulated periodically as the scatterer rotates, and contains a discrete spectrum of frequency components.

The scattered spectrum is predicted through electromagnetic field theory. The theory has been developed such as to exploit the angular periodicities of the ensembles and thereby reduce the computational load by a considerable factor.

The spectrum consists of lines separated by the "body rate", i.e. the rate of rotation multiplied by the number of bodies in the rotating ensemble The total bandwidth is several times greater than that for the rotating ensemble in free-space, due to electromagnetic interaction between the two ensembles.

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